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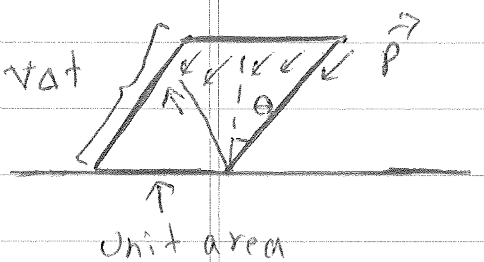
Thermal History of the Universe (Cont'd):

In addition to the number density n and energy density ρ, there are two other important quantities: pressure P and entropy density s.

First, let us find pressure for a general case. From definition, it is equal to the rate at which momentum is imparted by particles in the ensemble per unit area of a hypothetical surface:



$$\Delta p(\text{per particle}) = 2p\cos\theta$$

$$\Delta P(\text{per particles with momentum } \vec{p}) = 2p\cos\theta \; v\Delta t\cos\theta \times \left[\frac{1}{(2\pi)^3} f \; d^3p\right]$$

From special relativity we have:

$$E = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}, \quad p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow v = \frac{p}{E} = \frac{p}{\sqrt{p^2+m^2}}$$

$$\underset{\underset{P}{\shortparallel}}{d\left(\frac{\Delta P}{\Delta t}\right)} = \frac{1}{(2\pi)^3} \, 2p\cos^2\theta \, \frac{p}{\sqrt{p^2+m^2}} \, f \, d^3p$$

After using $d^3p = p^2 dp \sin\theta \, d\theta \, d\phi$, integration over the angular part gives:

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta \, d\phi = \frac{2\pi}{3}$$

Thus:

$$P = \frac{1}{2\pi^2} \int \frac{p^4}{3\sqrt{p^2+m^2}} f \, dp$$

Considering relativistic particles, $m=0$, we find:

$$P = \frac{1}{6\pi^2} \int p^3 f \, dp = \frac{1}{3} \rho$$

For a thermal distribution, this results in:

$$P = \left(g_B + \frac{7}{8} g_F\right) \frac{\pi^2}{90} T^4 \qquad (\mathrm{I})$$

Next, we find the entropy density s. From the first law of thermodynamics, we have:

$$T dS = dE + P dV \qquad E = \rho V, \; S = s V$$

This leads to:

$$Ts\,dV + TV\,ds = \rho\,dV + V\,d\rho + P\,dV \Rightarrow TV\,ds = (\rho + P)\,dV - Ts\,dV$$

$$+ V\,d\rho \Rightarrow ds = \left(\frac{\rho + P - Ts}{TV}\right) dV + \frac{V\,d\rho}{TV}$$

For relativistic particles in thermal equilibrium $\rho \propto T^4$, hence:

$$\frac{d\rho}{\rho} = \frac{4\,dT}{T} \Rightarrow d\rho = 4\frac{\rho}{T}\,dT$$

$$ds = \left(\frac{\rho + P - Ts}{TV}\right) dV + \frac{4\rho}{T^2}\,dT$$

Taking V and T as independent variables, we note that the entropy density s cannot have an explicit dependence on V. The reason being that S in an extensive quantity (hence a function of V) and not s. This can be intuitively understood as at a constant temperature s is independent from the volume. In consequence, $\left(\frac{\partial s}{\partial V}\right)_T = 0$, which results in:

$$\rho + P = Ts \Rightarrow s = \frac{\rho + P}{T} = \frac{4}{3}\frac{\rho}{T}$$

Therefore:

$$s = \left(g_B + \frac{7}{8} g_F\right) \frac{2\pi^2}{45} T^3 \quad (\mathrm{II})$$

At very early times, when temperature is very high, all of the elementary particles are in the relativistic regime. Therefore, expressions obtained for $n/\rho/P/s$ are valid for all of these particles at sufficiently early times. However, expansion of the universe results in a decrease in temperature ($T \propto a^{-1}$). For photons, which are massless, this does not change the situation as they always are relativistic. On the other hand, for massive particles the relativistic regime is not valid once $T \sim m$. As discussed before, there will be a transition from relativistic to non-relativistic regime in this case.

The expressions for the number density, energy density, and pressure of non-relativistic species are given by:

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right) \qquad \text{(III)}$$

$$\rho = mn \qquad P = nT \ll \rho$$

In the first equation, g denotes the number of non-relativistic degrees of freedom with mass m for bosons and fermions both. We note that, unlike the case of relativistic particles, the same expression is obtained for bosons and fermions.

An important difference between relativistic and non-relativistic cases arises because of the exponential suppression in the latter case. If relativistic particles start/reach thermal equilibrium at temperature T, they will always remain in equilibrium in an expanding universe. This can be understood qualitatively as $n_{eq} \propto T^3 \propto a^{-3}$, which is the same as dilution due to expansion. However, for non-relativistic particles, $n_{eq} \propto \exp\left(-\frac{m}{T}\right) \propto \exp(-am)$, which is much more suppressed than dilution due to expansion $\propto a^{-3}$. This implies that expansion alone cannot maintain thermal equilibrium of non-relativistic particles. Other physical processes are needed to decrease the number density fast enough so that it can follow the exponentially suppressed value of n_{eq}.

As an example, consider electrons. Until the time of recombination $t_{rec} \simeq 400{,}000$ yr, electrons were not bound to protons. At t_{rec} the temperature of the universe was $T_{rec} \sim 0.3$ eV. The number density of photons at this time is $n_{\gamma, rec} \propto T_{rec}^3$. If electrons were had a thermal equilibrium distribution too, then their number density would be $n_{e, rec} \propto (m_e T_{rec})^{3/2} \exp\left(-\frac{m_e}{T_{rec}}\right)$. Considering that $m_e \simeq 0.5$ MeV, this is an extremely small number, implying $\frac{n_e}{n_\gamma} \ll 10^{-9}$. As mentioned before, $\frac{n_e}{n_\gamma} \sim O(10^{-9})$ is inferred from the observations. This implies that electrons did not keep their thermal equilibrium until t_{rec}. Therefore, processes like $e^- e^+ \to \gamma\gamma$ that decrease the number of electrons were rendered ineffective long before the recombination. This is the reason why $\frac{n_e}{n_\gamma}$ is much larger than what we would have if thermal equilibrium was maintained until t_{rec}.

Processes lik $e^- e^+ \to \gamma\gamma$ can becom ineffective in two ways. First, if they are initially efficient in keeping n_e at its equilibrium value, they soon become ineffective as a result of the exponential suppression in $n_{e,eq}$. The process is self limiting in that it becomes more and more difficult for electrons and positrons to find each other and annihilate. Eventually, $n_{e,eq}$ becomes so small that no significant annihilation can take place within a Hubble time H^{-1}. A second possibility is that the process is forbidden altogether because of some conservation law. For example, if the number of electrons is larger than that of positrons, only electrons will be left after sufficient annihilation. From then on, there will be no further annihilation since a process like $e^- e^- \to \gamma\gamma$ is forbidden by conservation of electric charge. A typical annihilation process is rendered ineffective in an expanding universe at the earlier time imposed by the two possible mechanism. As we will see later, the asymmetry in the number of electrons and protons is the main factor that makes the annihilation process, which can reduce n_e, inefficient.

Once the annihilation processes like this are inefficient, the number density of that species changes only because of expansion. From this point on, it will follow $n \propto a^{-3}$. As a result, when normalized to the entropy density in relativistic degrees of freedom $s \propto T^3$ (which is redshifted $\propto a^{-3}$), it yields a quantity $\frac{n}{s}$ that is a constant during expansion of the universe. This quantity is called "Comoving number density" of that species. The comoving number density, after annihilation processes are rendered inefficient, can only change if the entropy density s changes. This can happen when there is an out-of-equilibrium process that results in entropy increase.

As we will discuss in detail, the observed density of baryons and dark matter in the universe is set by some annihilation processes going out of equilibrium. The processes that are relevant for relic density of baryons and dark matter are very different and go out of equilibrium at different times. Nevertheless, inefficiency of some annihilation process in an expanding universe, is responsible for both of these densities.